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Analytic Expressions for the Smear Due to Nonlinear Multipoles

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I. INTRODUCTION

The linear aperture is a region around the axis of the magnets in a collider ring where the particle motion is sufficiently linear. Such a region is necessary to allow for stable and efficient beam operations. The linear aperture is also regarded as the prime requirement in choosing the inner diameter of the magnet coil package. A smaller coil size implies a lower magnet cost.

For truly linear motion, the particle trajectory in the phase space at a certain location along the ring maps out a perfect ellipse which is an invariant. In the presence of nonlinearities, however, the trajectory fluctuates about the ellipse from turn to turn. The rms fractional value of this fluctuation is called the *single particle smear*. Based on past accelerator experience,[1,2] the linear aperture of the SSC is defined quantitatively[3] as the region within which the smear is less than 6.4% and the on-momentum tune shift with amplitude is less than 0.005.

In experiment E778 performed at the Fermilab Tevatron, the multiparticle smear was measured[4] for various sextupole excitation currents. The results appeared to agree excellently with multiparticle simulations. Single-particle tracking was then performed with exactly the same machine inputs as the multiparticle simulations to convert the observed smear of the beam to that of a single particle. It would be very useful that the single particle smear can be computed directly from the lattice. This is because such computation can tell us immediately whether the multipole errors in a particular sample dipole and the particular distributions of correction multipoles are acceptable or not without resorting to extensive tracking. Such possible screening procedure for random multipole field errors has been attempted by Peggs, Furman, and Chao.[5]

In this paper, analytical formulæ are presented for the computation of smear due to both field errors and correction multipole insertions. Specifically the one- and two-degree-of-freedom expressions for the smear due to sextupoles and octupoles are given first. In this particular case we show that the smear has a very simple expression in terms of the norms of Collins' distortion functions.[6] It will be shown later that this is not the case upon the addition of higher multipole terms. The reason is that there is in general more than one multipole contributing to the same harmonic component and hence their contribution is not separable. Next, a generalized formula for the horizontal smear due to all multipoles is derived. Since horizontal and vertical coupling is usually minimized during actual machine operations, the formula for horizontal smear should be adequate for all practical purposes. In these derivations the periodic structure of the multipoles in the ring is taken into account. Analytic expressions for the smear have also been derived by Forest[7] but in the complicated hard-to-follow Lie algebra language. Our analytic expressions are simple.

Finally, a number of applications demonstrate the usefulness of these calculations. In particular, the smear as calculated in this formalism, is compared with the smear extracted from experimental observations during E778, for a variety of conditions. Results from tracking calculations are also presented and comparisons are made. Moreover, the smear introduced in the Tevatron due to random and systematic multipole errors in the dipole magnets is calculated. The same calculation is repeated for the SSC ring. Furthermore, the calculation of the smear in the SSC is done after the insertion of correction elements for random multipole errors. The lumped correction scheme chosen for this calculation is due to Neuffer.[8] Various cases are examined and comparisons with tracking results from Sun and Talman,[9] and other analytical calculations due to Forest and Peterson[10] are presented.

II. SMEAR DUE TO NORMAL SEXTUPOLES

One-Degree-of-Freedom Calculation

Consider the situation of having only sextupoles in the ring. For first order perturbation, the distortion of the horizontal particle amplitude $\mathcal{A}_x$ at phase advance ψ_x is given by[11]

$$\begin{aligned}\delta\mathcal{A}_x(\psi_x) = \mathcal{A}_x^2\{[A_1(\psi_x)\sin\varphi_x - B_1(\psi_x)\cos\varphi_x] \\ + [A_3(\psi_x)\sin 3\varphi_x - B_3(\psi_x)\cos 3\varphi_x]\}\ ,\end{aligned} \tag{2.1}$$

where φ_x is the instantaneous betatron phase such that

$$x = \mathcal{A}_x \cos\varphi_x \tag{2.2}$$

and

$$x' = -\mathcal{A}_x \sin\varphi_x, \tag{2.3}$$

and the phase advance ψ_x is defined by

$$\psi_x(s) = \int_0^s \frac{ds'}{\beta_x(s')}. \tag{2.4}$$

Here B_1, A_1, B_3, and A_3 are the Collins' distortion functions defined by

$$\begin{aligned}
B_1(\psi_x) &= \frac{1}{2\sin\pi\nu_x}\sum_k \frac{S_k^{(2)}}{4}\cos(\psi'_{xk}-\psi_x-\pi\nu_x)\ , \\
A_1(\psi_x) &= \frac{1}{2\sin\pi\nu_x}\sum_k \frac{S_k^{(2)}}{4}\sin(\psi'_{xk}-\psi_x-\pi\nu_x)\ , \\
B_3(3\psi_x) &= \frac{1}{2\sin3\pi\nu_x}\sum_k \frac{S_k^{(2)}}{4}\cos3(\psi'_{xk}-\psi_x-\pi\nu_x)\ , \\
A_3(3\psi_x) &= \frac{1}{2\sin3\pi\nu_x}\sum_k \frac{S_k^{(2)}}{4}\sin3(\psi'_{xk}-\psi_x-\pi\nu_x)\ .
\end{aligned} \tag{2.5}$$

The summations over k above, are over each sextupole located at the 'modified' phase advance ψ'_{xk}, which is related to the usual Floquet phase ψ_{xk} by

$$\psi'_{xk} = \begin{cases} \psi_{xk} & \text{if}\ \ \psi_{xk} \geq \psi_x\ , \\ \psi_{xk}+2\pi\nu_x & \text{if}\ \ \psi_{xk} < \psi_x\ , \end{cases} \tag{2.6}$$

depending on whether it is upstream or downstream of the observation point ψ_x.

The horizontal amplitude $\mathcal{A}_x$ has been normalized so that it is a constant of motion (or the Courant-Snyder ellipse is a circle) for the perfectly linear machine. It is related to the horizontal emittance ϵ_x by

$$\epsilon_x = \frac{\pi\mathcal{A}_x^2}{\beta_0}\ , \tag{2.7}$$

where β_0 is a reference length specially introduced so that the amplitude $\mathcal{A}_x$ retains the dimension of length. The normalized vertical amplitude $\mathcal{A}_y$ has been set to zero.

The single particle smear at ψ_x is defined as

$$S_X(\psi_x) = \frac{\sqrt{\langle \mathcal{A}_x^2 \rangle - \langle \mathcal{A}_x \rangle^2}}{\langle \mathcal{A}_x \rangle} \tag{2.8}$$

or

$$S_X(\psi_x) = \left(\frac{\langle (\delta \mathcal{A}_x)^2 \rangle}{\mathcal{A}_x^2} \right)^{1/2} , \tag{2.9}$$

where $\langle \ \rangle$ denotes the average over *many* turns, or, equivalently over the instantaneous betatron phase φ_x. From Eq. (2.1), we get immediately

$$S_X^2(\psi_x) = \frac{1}{2} \mathcal{A}_x^2 \left\{ A_3^2(\psi_x) + B_3^2(\psi_x) + A_1^2(\psi_x) + B_1^2(\psi_x) \right\} . \tag{2.10}$$

If we consider the distortion functions as vectors

$$R_1^{(2)}(\psi_x) = \begin{pmatrix} B_1(\psi_x) \\ A_1(\psi_x) \end{pmatrix} \qquad R_3^{(2)}(\psi_x) = \begin{pmatrix} B_3(\psi_x) \\ A_3(\psi_x) \end{pmatrix} , \tag{2.11}$$

the smear is just related to the norms of these vectors by

$$S_X^2(\psi_x) = \frac{1}{2} \mathcal{A}_x^2 \left\{ |R_1^{(2)}|^2 + |R_3^{(2)}|^2 \right\}_{\psi_x} . \tag{2.12}$$

It is a well-known property of the distortion functions that the distortion functions at another point $\psi + \Delta\psi$ downstream are given by the vectors $R_1^{(2)}$ and $R_3^{(2)}$ rotated through angles $\Delta\psi$ and $3\Delta\psi$ respectively if there is no sextupole between the two points:

$$\begin{pmatrix} B_p \\ A_p \end{pmatrix}_{\psi+\Delta\psi} = \begin{pmatrix} \cos p\Delta\psi & \sin p\Delta\psi \\ -\sin p\Delta\psi & \cos p\Delta\psi \end{pmatrix} \begin{pmatrix} B_p \\ A_p \end{pmatrix}_{\psi} , \tag{2.13}$$

where p stands for 1 or 3. In passing through a *thin* sextupole of length $L \to 0$ and strength

$$\mathcal{S}^{(2)} = \lim_{L \to 0} \left[\left(\frac{\beta_x^3}{\beta_0} \right)^{1/2} \frac{B_y'' L}{2(B\rho)} \right] , \tag{2.14}$$

with horizontal betatron function β_x and particle's magnetic rigidity $(B\rho)$, the B_p's are continuous while the A_p's jump by an amount $\mathcal{S}^{(2)}/4$. Thus the smear will be a constant between two sextupoles but will have a jump when a sextupole is crossed. This is demonstrated in Fig. (1) which is obtained by plotting the smear as given by Eq. (2.12)

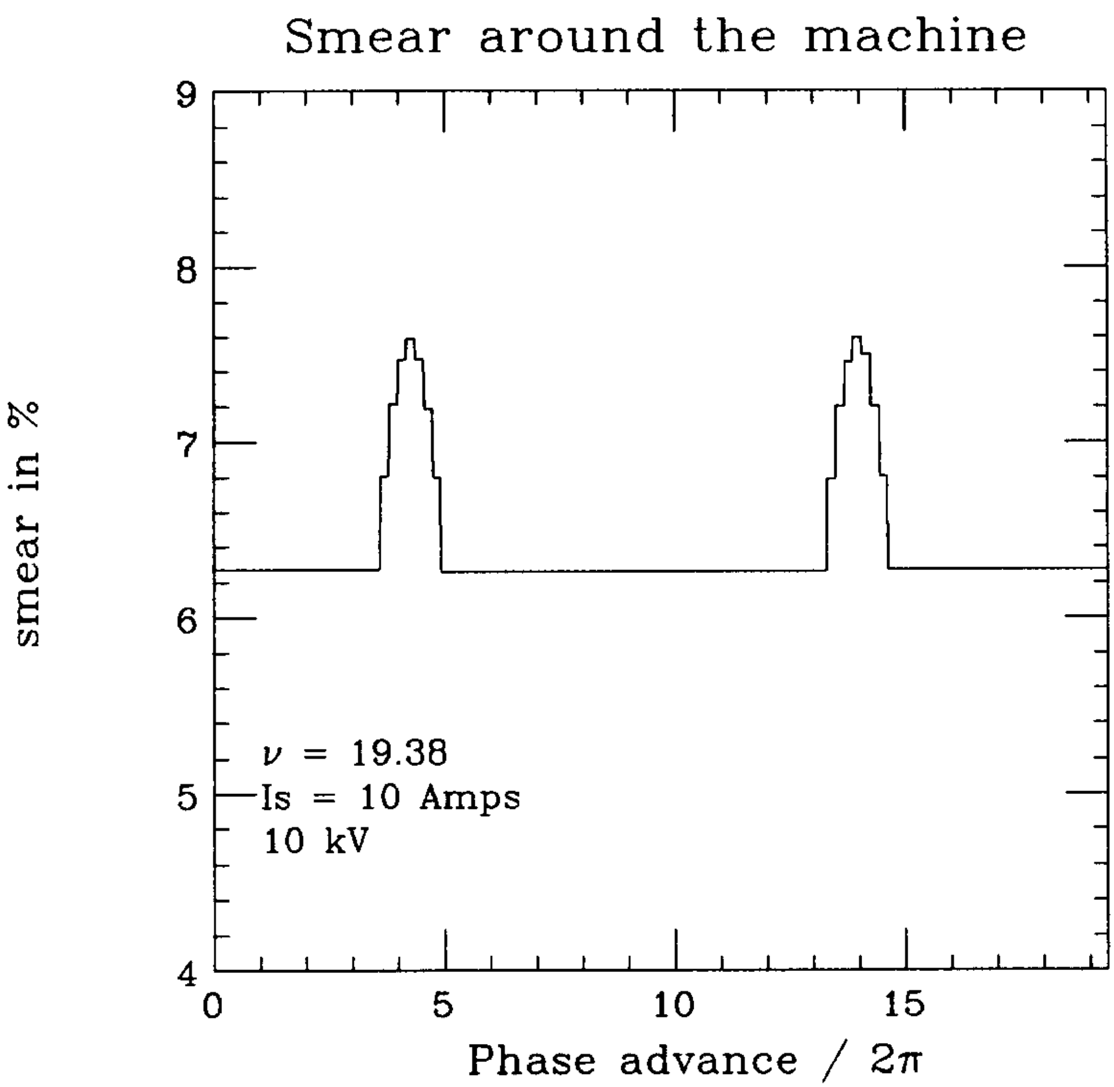

Figure 1: Smear versus phase advance, around the machine, as predicted from perturbation calculation.

as a function of the phase advance around the Tevatron during the performance of E778. Sixteen sextupoles clustered in two groups of eight located at phase advances of approximately $4.5 \times 2\pi$ and $14.5 \times 2\pi$ cause these jumps in the smear. In the special situation of having only one sextupole in the ring, the smear becomes a constant of motion.

Further insight can be obtained from the expression (2.12) for the smear in terms of phases and sextupole strengths. The definition of the distortion functions (2.5) implies that

$$B_1^2(\psi_x) + A_1^2(\psi_x) = \frac{1}{64 \sin^2 \pi\nu_x} \sum_{k,k'} \mathcal{S}_k^{(2)} \mathcal{S}_{k'}^{(2)} \cos(\psi'_{xk} - \psi'_{xk'}) \ , \tag{2.15}$$

or

$$|R_1^{(2)}(\psi_x)| = \frac{1}{8|\sin \pi\nu_x|} \left| \sum_k \mathcal{S}_k^{(2)} e^{i\psi'_{xk}} \right| \ . \tag{2.16}$$

Similarly,

$$|R_3^{(2)}(\psi_x)| = \frac{1}{8|\sin 3\pi\nu_x|} \left| \sum_k \mathcal{S}_k^{(2)} e^{i3\psi'_{xk}} \right| \ . \tag{2.17}$$

Therefore, if we want the horizontal smear to vanish at a particular point ψ, what we need is to arrange the sextupoles so that

$$\sum_k \mathcal{S}_k^{(2)} e^{i\psi'_k} = 0 \qquad \text{and} \qquad \sum_k \mathcal{S}_k^{(2)} e^{i3\psi'_k} = 0 \ . \tag{2.18}$$

If the sextupoles are random errors in dipoles say, then we have

$$R_1^{(2)^2} + R_3^{(2)^2} = \frac{1}{64} \left(\frac{1}{\sin^2 \pi\nu_x} + \frac{1}{\sin^2 3\pi\nu_x} \right) \sum_k \langle \mathcal{S}_k^{(2)\,2} \rangle \ , \tag{2.19}$$

where $\langle \mathcal{S}_k^{(2)\,2} \rangle$ is the variance of the sextupole error at location k. Note that for purely random errors, the smear is identically the same at every point around the ring. Thus, these errors cannot be corrected in any way. However, in a machine like the Tevatron, every dipole has been measured. In other words, all the "random errors" at the dipoles are actually known. As a result, they can be corrected using correctors. In Chapter V we present examples of how one can calculate the smear due to random sextupole errors in realistic cases.

Two-Degree-of-Freedom Calculation

In two degrees of freedom the distortion of the horizontal amplitude $\mathcal{A}_x$ at phase advance ψ_x, to first order in the sextupole strength, is given by[11]

$$\begin{aligned}\delta\mathcal{A}_x = {} & \mathcal{A}_x^2[(A_1\sin\varphi_x - B_1\cos\varphi_x) + (A_3\sin 3\varphi_x - B_3\cos 3\varphi_x)] \\ & -\mathcal{A}_y^2[2(\bar{A}\sin\varphi_x - \bar{B}\cos\varphi_x) + (A_s\sin\varphi_+ - B_s\cos\varphi_+) \\ & \quad - (A_d\sin\varphi_- - B_d\cos\varphi_-)] \ . \end{aligned} \tag{2.20}$$

The distortion of the vertical amplitude $\mathcal{A}_y$ is given by

$$\delta\mathcal{A}_y = -2\mathcal{A}_x\mathcal{A}_y[(A_s\sin\varphi_+ - B_s\cos\varphi_+) + (A_d\sin\varphi_- - B_d\cos\varphi_-)] \ . \tag{2.21}$$

The distortion functions A_1, B_1, A_3, B_3 are given by Eq. (2.5) while B_s, A_s, B_d, A_d, $\bar{B}$, $\bar{A}$ are given by

$$\begin{aligned} B_s(\psi_+) &= \frac{1}{2\sin\pi\nu_+}\sum_k \frac{\bar{S}_k^{(2)}}{4}\cos(\psi'_{+k} - \psi_+ - \pi\nu_+) \ , \\ A_s(\psi_+) &= \frac{1}{2\sin\pi\nu_+}\sum_k \frac{\bar{S}_k^{(2)}}{4}\sin(\psi'_{+k} - \psi_+ - \pi\nu_+) \ , \\ B_d(\psi_-) &= \frac{1}{2\sin\pi\nu_-}\sum_k \frac{\bar{S}_k^{(2)}}{4}\cos(\psi'_{-k} - \psi_- - \pi\nu_-) \ , \\ A_d(\psi_-) &= \frac{1}{2\sin\pi\nu_-}\sum_k \frac{\bar{S}_k^{(2)}}{4}\sin(\psi'_{-k} - \psi_- - \pi\nu_-) \ , \\ \bar{B}(\psi_x) &= \frac{1}{2\sin\pi\nu_x}\sum_k \frac{\bar{S}_k^{(2)}}{4}\cos(\psi'_{xk} - \psi_x - \pi\nu_x) \ , \\ \bar{A}(\psi_x) &= \frac{1}{2\sin\pi\nu_x}\sum_k \frac{\bar{S}_k^{(2)}}{4}\sin(\psi'_{xk} - \psi_x - \pi\nu_x) \ , \end{aligned} \tag{2.22}$$

where $\psi_\pm = 2\psi_y \pm \psi_x$ and $\nu_\pm = 2\nu_y \pm \nu_x$. Again here ψ'_{xk} and ψ'_{yk} are the modified phase advances defined in Eq. (2.6). The sextupole strength $\bar{S}^{(2)}$ is defined by

$$\bar{S}^{(2)} = \lim_{L\to 0}\left[\left(\frac{\beta_x\beta_y^2}{\beta_0}\right)^{1/2}\frac{B_y''L}{2(B\rho)}\right] . \tag{2.23}$$

In two degrees of freedom one can define three different kinds of smear:

$$S_{XX} = \left(\frac{\langle (\delta \mathcal{A}_x)^2 \rangle}{\mathcal{A}_x^2} \right)^{1/2} , \tag{2.24}$$

$$S_{YY} = \left(\frac{\langle (\delta \mathcal{A}_y)^2 \rangle}{\mathcal{A}_y^2} \right)^{1/2} , \tag{2.25}$$

and

$$S_{XY} = \left(\frac{\langle (\delta \mathcal{A}_x \delta \mathcal{A}_y) \rangle}{\mathcal{A}_x \mathcal{A}_y} \right)^{1/2} , \tag{2.26}$$

where $\langle\ \rangle$ denotes again the average over the instantaneous betatron phases φ_x and φ_y. Using Eqs. (2.20) and (2.21) one can express the three smears in terms of the Collins' distortion functions as follows

$$\begin{aligned} S_{XX}^2 &= \frac{1}{2} \mathcal{A}_x^2 (A_1^2 + B_1^2 + A_3^2 + B_3^2) \\ &+ \frac{1}{2} \frac{\mathcal{A}_y^2}{\mathcal{A}_x^2} [A_s^2 + B_s^2 + A_d^2 + B_d^2 + 4(\bar{A}^2 + \bar{B}^2)] - 2\mathcal{A}_y^2 (A_1 \bar{A} + B_1 \bar{B}) \ , \quad (2.27) \\ S_{YY}^2 &= 2\mathcal{A}_x^2 (A_s^2 + B_s^2 + A_d^2 + B_d^2) \quad (2.28) \\ S_{XY}^2 &= \mathcal{A}_y^2 (A_s^2 + B_s^2 - A_d^2 - B_d^2) \ . \quad (2.29) \end{aligned}$$

If we consider the distortion functions as vectors, similarly to the one-degree-of-freedom case,

$$R_s(\psi_+) = \begin{pmatrix} B_s(\psi_+) \\ A_s(\psi_+) \end{pmatrix} , \quad R_d(\psi_-) = \begin{pmatrix} B_d(\psi_-) \\ A_d(\psi_-) \end{pmatrix} , \quad \bar{R}(\psi_x) = \begin{pmatrix} \bar{B}(\psi_x) \\ \bar{A}(\psi_x) \end{pmatrix} , \tag{2.30}$$

the three smears are expressed in terms of the norms of these vectors as follows

$$S_{XX}^2 = \frac{1}{2} \mathcal{A}_x^2 \left\{ |R_1|^2 + |R_3|^2 \right\}_{\psi_x} + \frac{1}{2} \frac{\mathcal{A}_y^4}{\mathcal{A}_x^2} \left\{ |R_s|^2 + |R_d|^2 + 4|\bar{R}|^2 \right\}_{\psi_x} - 2\mathcal{A}_y^2 (R_1, \bar{R}) , \tag{2.31}$$

$$S_{YY}^2 = 2\mathcal{A}_x^2 \left\{ |R_s|^2 + |R_d|^2 \right\}_{\psi_x} , \tag{2.32}$$

and

$$S_{XY}^2 = \mathcal{A}_y^2 \left\{ |R_s|^2 - |R_d|^2 \right\}_{\psi_x} . \tag{2.33}$$

In the above $(R_1, \bar{R})$ denotes the inner product of the vectors R_1 and $\bar{R}$.

From Eq. (2.22) we have

$$A_s^2 + B_s^2 = \frac{1}{64\sin^2\pi\nu_+}\sum_{k,k'}\bar{S}_k^{(2)}\bar{S}_{k'}^{(2)}\cos(\psi'_{+k} - \psi'_{+k'}) \ , \tag{2.34}$$

or

$$|R_s| = \frac{1}{8|\sin\pi\nu_+|}\left|\sum_k \bar{S}_k^{(2)} e^{i\psi'_{+k}}\right| . \tag{2.35}$$

Similarly

$$|R_d| = \frac{1}{8|\sin\pi\nu_-|}\left|\sum_k \bar{S}_k^{(2)} e^{i\psi'_{-k}}\right| , \tag{2.36}$$

$$|\bar{R}| = \frac{1}{8|\sin\pi\nu_x|}\left|\sum_k \bar{S}_k^{(2)} e^{i\psi'_{xk}}\right| , \tag{2.37}$$

and

$$(R_1, \bar{R}) = \frac{1}{64\sin^2\pi\nu_x}\sum_{k,k'} S_k^{(2)}\bar{S}_{k'}^{(2)}\cos\left(\psi'_{xk} - \psi'_{xk'}\right) . \tag{2.38}$$

Hence (2.27), (2.28) and (2.29) become

$$\begin{aligned} S_{XX}^2 = {} & \frac{1}{2}\,\frac{\mathcal{A}_x^2}{64}\left\{\frac{|\sum_k S_k^{(2)} e^{i\psi'_k}|^2}{\sin^2\pi\nu_x} + \frac{|\sum_k S_k^{(2)} e^{i3\psi'_k}|^2}{\sin^2 3\pi\nu_x}\right\} \\ & + \frac{1}{2}\,\frac{\mathcal{A}_y^4}{64\mathcal{A}_x^2}\left\{\frac{|\sum_k \bar{S}_k^{(2)} e^{i\psi'_{+k}}|^2}{\sin^2\pi\nu_+} + \frac{|\sum_k \bar{S}_k^{(2)} e^{i\psi'_{-k}}|^2}{\sin^2\pi\nu_-} + \frac{4|\sum_k \bar{S}_k^{(2)} e^{i\psi'_{xk}}|^2}{\sin^2\pi\nu_x}\right\} \\ & - \frac{\mathcal{A}_y^2}{32}\,\frac{1}{\sin^2\pi\nu_x}\sum_{k,k'} S_k^{(2)}\bar{S}_{k'}^{(2)}\cos\left(\psi'_{xk} - \psi'_{xk'}\right) \ , \end{aligned} \tag{2.39}$$

$$S_{YY}^2 = \frac{\mathcal{A}_x^2}{32}\left\{\frac{|\sum_k \bar{S}_k^{(2)} e^{i\psi'_{+k}}|^2}{\sin^2\pi\nu_+} + \frac{|\sum_k \bar{S}_k^{(2)} e^{i\psi'_{-k}}|^2}{\sin^2\pi\nu_-}\right\} \tag{2.40}$$

and

$$S_{XY}^2 = \frac{\mathcal{A}_y^2}{64}\left\{\frac{|\sum_k \bar{S}_k^{(2)} e^{i\psi'_{+k}}|^2}{\sin^2\pi\nu_+} - \frac{|\sum_k \bar{S}_k^{(2)} e^{i\psi'_{-k}}|^2}{\sin^2\pi\nu_-}\right\} . \tag{2.41}$$

III. SMEAR DUE TO NORMAL OCTUPOLES

One-Degree-of-Freedom Calculation

The distortion of the horizontal amplitude $\mathcal{A}_x$ due to normal octupoles, is given, to first order in the octupole strength, by

$$\delta\mathcal{A}_x = \mathcal{A}_x^3[(A_1 \sin 4\varphi_x - B_1 \cos 4\varphi_x) + 2(A_2 \sin 2\varphi_x - B_2 \cos 2\varphi_x)] \ , \tag{3.1}$$

where φ_x is the instantaneous betatron phase given through Eqs. (2.2) and (2.3) and A_1, B_1, A_2, B_2 are the Collins' distortion functions defined by

$$B_1(\psi_x) = \frac{1}{2\sin 4\pi\nu_x} \sum_k \frac{\underline{S}_k^{(3)}}{8} \cos 4(\psi'_{xk} - \psi_x - \pi\nu_x) \ ,$$

$$A_1(\psi_x) = \frac{1}{2\sin 4\pi\nu_x} \sum_k \frac{\underline{S}_k^{(3)}}{8} \sin 4(\psi'_{xk} - \psi_x - \pi\nu_x) \ ,$$

$$B_2(\psi_x) = \frac{1}{2\sin 2\pi\nu_x} \sum_k \frac{\underline{S}_k^{(3)}}{8} \cos 2(\psi'_{xk} - \psi_x - \pi\nu_x) \ ,$$

$$A_2(\psi_x) = \frac{1}{2\sin 2\pi\nu_x} \sum_k \frac{\underline{S}_k^{(3)}}{8} \sin 2(\psi'_{xk} - \psi_x - \pi\nu_x) \ . \tag{3.2}$$

The octupole strength $\underline{S}^{(3)}$ is defined by

$$\underline{S}^{(3)} = \lim_{L\to 0}\left[\left(\frac{\beta_x^2}{\beta_0}\right)\frac{B_y'''L}{6(B\rho)}\right] . \tag{3.3}$$

Also ψ'_x denotes the modified phase advance as before.

Hence the horizontal smear given by Eq. (2.9) is

$$S_X^2 = \frac{1}{2}\mathcal{A}_x^4\left\{(A_1^2 + B_1^2) + 4(A_2^2 + B_2^2)\right\} \ , \tag{3.4}$$

or

$$S_X^2 = \frac{1}{2}\mathcal{A}_x^4\left\{|R_1^{(3)}|^2 + 4|R_2^{(3)}|^2\right\} \ , \tag{3.5}$$

where

$$|R_1^{(3)}| = \frac{1}{16|\sin 4\pi\nu_x|}\left|\sum_k \underline{S}_k^{(3)} e^{i4\psi'_{xk}}\right| \tag{3.6}$$

and

$$|R_2^{(3)}| = \frac{1}{16|\sin 2\pi\nu_x|}\left|\sum_k \mathcal{S}_k^{(3)} e^{i2\psi'_{xk}}\right| . \tag{3.7}$$

If the horizontal smear is due to both sextupoles and octupoles then it is given by the sum of Eqs (2.12) and (3.5), or

$$S_X^2 = \frac{1}{2}\mathcal{A}_x^2\left\{|R_1^{(2)}|^2 + |R_3^{(2)}|^2\right\} + \frac{1}{2}\mathcal{A}_x^4\left\{|R_1^{(3)}|^2 + 4|R_2^{(3)}|^2\right\} . \tag{3.8}$$

This expression can be generalized to include the effects of all higher multipoles. This is the subject of the next section. Before that though, we shall derive expressions for the two-degree-of-freedom smears due to octupoles.

Two-Degree-of-Freedom Calculation

In two degrees of freedom the distortions of the horizontal and vertical amplitudes, $\mathcal{A}_x$ and $\mathcal{A}_y$ respectively, are given by

$$\begin{aligned}\delta\mathcal{A}_x =\ & \mathcal{A}_x^3[(A_1\sin 4\varphi_x - B_1\cos 4\varphi_x) + 2(A_2\sin 2\varphi_x - B_2\cos 2\varphi_x)]\\ & -3\mathcal{A}_x\mathcal{A}_y^2[2(A_5\sin 2\varphi_x - B_5\cos 2\varphi_x) + (A_3\sin 2\varphi_+ - B_3\cos 2\varphi_+)\\ & +(A_4\sin 2\varphi_- - B_4\cos 2\varphi_-)],\end{aligned} \tag{3.9}$$

$$\begin{aligned}\delta\mathcal{A}_y =\ & -3\mathcal{A}_x^2\mathcal{A}_y[2(A_6\sin 2\varphi_y - B_6\cos 2\varphi_y) + (A_3\sin 2\varphi_+ - B_3\cos 2\varphi_+)\\ & -(A_4\sin 2\varphi_- - B_4\cos 2\varphi_-)]\\ & +\mathcal{A}_y^3[2(A_8\sin 2\varphi_y - B_8\cos 2\varphi_y) + (A_7\sin 4\varphi_y - B_7\cos 4\varphi_y)] .\end{aligned} \tag{3.10}$$

The distortion functions A_1, B_1, A_2, B_2 are given by Eqs. (3.2) while A_3, B_3, A_4, B_4, A_5, B_5, A_6, B_6, A_7, B_7, and A_8, B_8 are given by

$$B_3(\psi_+) = \frac{1}{2\sin 2\pi\nu_+}\sum_k \frac{\mathcal{S}_k^{(3)}}{8}\cos 2(\psi'_{+k} - \psi_+ - \pi\nu_+) ,$$

$$A_3(\psi_+) = \frac{1}{2\sin 2\pi\nu_+}\sum_k \frac{\mathcal{S}_k^{(3)}}{8}\sin 2(\psi'_{+k} - \psi_+ - \pi\nu_+) ,$$

$$B_4(\psi_-) = \frac{1}{2\sin 2\pi\nu_-}\sum_k \frac{\mathcal{S}_k^{(3)}}{8}\cos 2(\psi'_{-k} - \psi_- - \pi\nu_-) ,$$

$$A_4(\psi_-) = \frac{1}{2\sin 2\pi\nu_-} \sum_k \frac{\mathcal{S}_k^{(3)}}{8} \sin 2(\psi'_{-k} - \psi_- - \pi\nu_-) \ ,$$

$$B_5(\psi_x) = \frac{1}{2\sin 2\pi\nu_x} \sum_k \frac{\mathcal{S}_k^{(3)}}{8} \cos 2(\psi'_{xk} - \psi_x - \pi\nu_x) \ ,$$

$$A_5(\psi_x) = \frac{1}{2\sin 2\pi\nu_x} \sum_k \frac{\mathcal{S}_k^{(3)}}{8} \sin 2(\psi'_{xk} - \psi_x - \pi\nu_x) \ ,$$

$$B_6(\psi_y) = \frac{1}{2\sin 2\pi\nu_y} \sum_k \frac{\mathcal{S}_k^{(3)}}{8} \cos 2(\psi'_{yk} - \psi_y - \pi\nu_y) \ ,$$

$$A_6(\psi_y) = \frac{1}{2\sin 2\pi\nu_y} \sum_k \frac{\mathcal{S}_k^{(3)}}{8} \sin 2(\psi'_{yk} - \psi_y - \pi\nu_y) \ ,$$

$$B_7(\psi_y) = \frac{1}{2\sin 4\pi\nu_y} \sum_k \frac{\bar{\mathcal{S}}_k^{(3)}}{8} \cos 4(\psi'_{yk} - \psi_y - \pi\nu_y) \ ,$$

$$A_7(\psi_y) = \frac{1}{2\sin 4\pi\nu_y} \sum_k \frac{\bar{\mathcal{S}}_k^{(3)}}{8} \sin 4(\psi'_{yk} - \psi_y - \pi\nu_y) \ ,$$

$$B_8(\psi_y) = \frac{1}{2\sin 2\pi\nu_y} \sum_k \frac{\bar{\mathcal{S}}_k^{(3)}}{8} \cos 2(\psi'_{yk} - \psi_y - \pi\nu_y) \ ,$$

$$A_8(\psi_y) = \frac{1}{2\sin 2\pi\nu_y} \sum_k \frac{\bar{\mathcal{S}}_k^{(3)}}{8} \sin 2(\psi'_{yk} - \psi_y - \pi\nu_y) \ . \tag{3.11}$$

Here $\nu_\pm = \nu_x \pm \nu_y$ and $\psi_\pm = \psi_x \pm \psi_y$. The octupole strengths $\mathcal{S}^{(3)}$ and $\bar{\mathcal{S}}^{(3)}$ are defined by

$$\mathcal{S}^{(3)} = \lim_{L\to 0} \left[\left(\frac{\beta_x\beta_y}{\beta_0} \right) \frac{B_y'''L}{6(B\rho)} \right] , \tag{3.12}$$

and

$$\bar{\mathcal{S}}^{(3)} = \lim_{L\to 0} \left[\left(\frac{\beta_y^2}{\beta_0} \right) \frac{B_y'''L}{6(B\rho)} \right] . \tag{3.13}$$

Again ψ' denotes the modified phase advance. Then the three different smears given by (2.24), (2.25) and (2.26) are

$$S_{XX}^2 = \frac{1}{2}\mathcal{A}_x^4[(A_1^2+B_1^2)+4(A_2^2+B_2^2)] + \frac{9}{2}\mathcal{A}_y^4[4(A_5^2+B_5^2)+(A_3^2+B_3^2)+(A_4^2+B_4^2)]$$
$$- 12\mathcal{A}_x^2\mathcal{A}_y^2[A_2A_5+B_2B_5] \ , \tag{3.14}$$

$$S^2_{YY} = \frac{9}{2}\mathcal{A}_x^4[4(A_6^2+B_6^2) + (A_3^2+B_3^2) + (A_4^2+B_4^2)] + \frac{1}{2}\,\mathcal{A}_y^4[4(A_8^2+B_8^2) + (A_7^2+B_7^2)]$$
$$- 12\mathcal{A}_x^2\mathcal{A}_y^2[A_6A_8+B_6B_8]\ , \tag{3.15}$$

and

$$S^2_{XY} = \frac{9}{2}\mathcal{A}_x^2\mathcal{A}_y^2[(A_3^2+B_3^2) - (A_4^2+B_4^2)]. \tag{3.16}$$

We define, as before, the vectors $R_3^{(3)}$, $R_4^{(3)}$, $R_5^{(3)}$, $R_6^{(3)}$, $R_7^{(3)}$ and $R_8^{(3)}$ such that

$$R_3^{(3)}(\psi_+) = \begin{pmatrix} B_3(\psi_+) \\ A_3(\psi_+) \end{pmatrix}\ , \quad R_4^{(3)}(\psi_-) = \begin{pmatrix} B_4(\psi_-) \\ A_4(\psi_-) \end{pmatrix}\ , \quad R_5^{(3)}(\psi_x) = \begin{pmatrix} B_5(\psi_x) \\ A_5(\psi_x) \end{pmatrix}\ ,$$

$$R_6^{(3)}(\psi_y) = \begin{pmatrix} B_6(\psi_y) \\ A_6(\psi_y) \end{pmatrix}\ , \quad R_7^{(3)}(\psi_y) = \begin{pmatrix} B_7(\psi_y) \\ A_7(\psi_y) \end{pmatrix}\ , \quad R_8^{(3)}(\psi_y) = \begin{pmatrix} B_8(\psi_y) \\ A_8(\psi_y) \end{pmatrix}\ . \tag{3.17}$$

Then S^2_{XX}, S^2_{YY} and S^2_{XY} become

$$S^2_{XX} = \frac{1}{2}\,\mathcal{A}_x^4\left\{|R_1^{(3)}|^2 + 4|R_2^{(3)}|^2\right\} + \frac{9}{2}\mathcal{A}_y^4\left\{4|R_5^{(3)}|^2 + |R_3^{(3)}|^2 + |R_4^{(3)}|^2\right\}$$
$$- 12\mathcal{A}_x^2\mathcal{A}_y^2(R_2^{(3)}, R_5^{(3)})\ , \tag{3.18}$$

$$S^2_{YY} = \frac{9}{2}\mathcal{A}_x^4\left\{4|R_6^{(3)}|^2 + |R_3^{(3)}|^2 + |R_4^{(3)}|^2\right\} + \frac{1}{2}\,\mathcal{A}_y^4\left\{4|R_8^{(3)}|^2 + |R_7^{(3)}|^2\right\}$$
$$- 12\mathcal{A}_x^2\mathcal{A}_y^2(R_6^{(3)}, R_8^{(3)}) \tag{3.19}$$

and

$$S^2_{XY} = \frac{9}{2}\mathcal{A}_x^2\mathcal{A}_y^2\left\{|R_3^{(3)}|^2 - |R_4^{(3)}|^2\right\}\ . \tag{3.20}$$

Using the definition of the distortion functions we arrive at

$$|R_3^{(3)}| = \frac{1}{16|\sin 2\pi\nu_+|}\left|\sum_k \mathcal{S}_k^{(3)} e^{i2\psi'_{+k}}\right|\ , \tag{3.21}$$

$$|R_4^{(3)}| = \frac{1}{16|\sin 2\pi\nu_-|}\left|\sum_k \mathcal{S}_k^{(3)} e^{i2\psi'_{-k}}\right|\ , \tag{3.22}$$

$$|R_5^{(3)}| = \frac{1}{16|\sin 2\pi\nu_x|}\left|\sum_k \mathcal{S}_k^{(3)} e^{i2\psi'_{xk}}\right|\ , \tag{3.23}$$

$$|R_6^{(3)}| = \frac{1}{16|\sin 2\pi\nu_y|}\left|\sum_k \mathcal{S}_k^{(3)} e^{i2\psi'_{yk}}\right|\ , \tag{3.24}$$

$$|R_7^{(3)}| = \frac{1}{16|\sin 4\pi\nu_y|}\left|\sum_k \bar{\mathcal{S}}_k^{(3)} e^{i4\psi'_{yk}}\right| \ , \tag{3.25}$$

$$|R_8^{(3)}| = \frac{1}{16|\sin 2\pi\nu_y|}\left|\sum_k \bar{\mathcal{S}}_k^{(3)} e^{i2\psi'_{yk}}\right| \ . \tag{3.26}$$

Hence the three smears are given by

$$\begin{aligned} S_{XX}^2 = {} & \frac{1}{2}\frac{\mathcal{A}_x^4}{256}\left\{\frac{|\sum_k \underline{\mathcal{S}}_k^{(3)} e^{i4\psi'_{xk}}|^2}{\sin^2 4\pi\nu_x} + \frac{4|\sum_k \underline{\mathcal{S}}_k^{(3)} e^{i2\psi'_{xk}}|^2}{\sin^2 2\pi\nu_x}\right\} \\ & + \frac{9}{2}\frac{\mathcal{A}_y^4}{256}\left\{+\frac{4|\sum_k \mathcal{S}_k^{(3)} e^{i2\psi'_{xk}}|^2}{\sin^2 2\pi\nu_x} + \frac{|\sum_k \mathcal{S}_k^{(3)} e^{i2\psi'_{+k}}|^2}{\sin^2 2\pi\nu_+} + \frac{|\sum_k \mathcal{S}_k^{(3)} e^{i2\psi'_{-k}}|^2}{\sin^2 2\pi\nu_-}\right\} \\ & - 12\frac{\mathcal{A}_x^2\mathcal{A}_y^2}{256}\frac{1}{\sin^2 2\pi\nu_x}\sum_{k,k'} \underline{\mathcal{S}}_k^{(3)} \mathcal{S}_{k'}^{(3)} \cos 2(\psi'_{xk} - \psi'_{xk'}) \ , \end{aligned} \tag{3.27}$$

$$\begin{aligned} S_{YY}^2 = {} & \frac{9}{2}\frac{\mathcal{A}_x^4}{256}\left\{\frac{4|\sum_k \mathcal{S}_k^{(3)} e^{i2\psi'_{yk}}|^2}{\sin^2 2\pi\nu_y} + \frac{|\sum_k \mathcal{S}_k^{(3)} e^{i2\psi'_{+k}}|^2}{\sin^2 2\pi\nu_+} + \frac{|\sum_k \mathcal{S}_k^{(3)} e^{i2\psi'_{-k}}|^2}{\sin^2 2\pi\nu_-}\right\} \\ & + \frac{1}{2}\frac{\mathcal{A}_y^4}{256}\left\{\frac{4|\sum_k \bar{\mathcal{S}}_k^{(3)} e^{i2\psi'_{yk}}|^2}{\sin^2 2\pi\nu_y} + \frac{|\sum_k \bar{\mathcal{S}}_k^{(3)} e^{i4\psi'_{yk}}|^2}{\sin^2 4\pi\nu_y}\right\} \\ & - 12\frac{\mathcal{A}_x^2\mathcal{A}_y^2}{256}\frac{1}{\sin^2 2\pi\nu_y}\sum_{k,k'} \mathcal{S}_k^{(3)} \bar{\mathcal{S}}_{k'}^{(3)} \cos 2(\psi'_{yk} - \psi'_{yk'}) \end{aligned} \tag{3.28}$$

and

$$S_{XY}^2 = \frac{9}{2}\frac{\mathcal{A}_x^2\mathcal{A}_y^2}{256}\left\{\frac{|\sum_k \mathcal{S}_k^{(3)} e^{i2\psi'_{+k}}|^2}{\sin^2 2\pi\nu_+} - \frac{|\sum_k \mathcal{S}_k^{(3)} e^{i2\psi'_{-k}}|^2}{\sin^2 2\pi\nu_-}\right\} \ . \tag{3.29}$$

IV. HORIZONTAL SMEAR DUE TO ALL MULTIPOLES

In this section, we want to derive a formula for the horizontal smear with the contributions from all higher multipoles without resorting to the use of distortion functions.

The irrotational magnetic flux density inside the beam pipe can be written in general as

$$B_y + iB_x = B_0\sum_{n=1}^{\infty}(b_n + ia_n)(x+iy)^n \ , \tag{4.1}$$

where b_n and a_n are the normal and skew multipole coefficients, respectively, of order $2(n+1)$. For example,

$$b_n = \frac{1}{n!B_0}\frac{\partial^n B_y}{\partial x^n} . \tag{4.2}$$

In above, the vertical bending magnetic flux density B_0 as well as the field gradients of the focussing F and D quads have been excluded. Thus, Eq. (4.1) contains the contributions of all field errors as well as other inserted correction multipoles only. Since we are concerned with the isolated horizontal phase space only, Eq. (4.1) simplifies to

$$B_y = B_0 \sum_{n=1}^{\infty} b_n x^n . \tag{4.3}$$

In terms of action-angle variables I and a, the motion of a beam particle is described by the Hamiltonian[12,13]

$$H = \nu I + \Delta H , \tag{4.4}$$

where

$$\Delta H = \sum_{n=1}^{\infty} \frac{RB_0 b_n}{(B\rho)} \left(\frac{\beta_x}{\beta_0}\right)^{\frac{n+1}{2}} \frac{x^{n+1}}{n+1} . \tag{4.5}$$

The derivation is given in the Appendix. The subscript x will be suppressed below for convenience. In the Hamiltonian, the independent variable is $\theta = s/R$, where s is the distance measured along the ideal designed closed orbit and R is the average radius of that orbit. The normalized horizontal displacement is given by

$$x = \mathcal{A}\cos[Q(\theta) + a] , \tag{4.6}$$

where the normalized amplitude is

$$\mathcal{A} = (2I\beta_0)^{1/2} , \tag{4.7}$$

and the function

$$Q(\theta) = \psi(\theta) - \nu\theta \tag{4.8}$$

is periodic in θ.

With the help of the trigonometric identities

$$\cos^{2m+1}\phi = \frac{1}{2^{2m}} \sum_{\ell=0}^{m} \binom{2m+1}{\ell} \cos(2m-2\ell+1)\phi ,$$

$$\cos^{2m}\phi = \frac{1}{2^{2m-1}} \sum_{\ell=0}^{m} \binom{2m}{\ell} \frac{\cos 2(m-\ell)\phi}{\delta_{\ell m}+1} \ , \tag{4.9}$$

for $m = 1,\ 2,\ 3,\ \dots$, Eq. (4.5), the higher multipole dependent part and field error part of the Hamiltonian, can be rewritten as

$$\Delta H = \sum_{m=1}^{\infty} \sum_{\ell=0}^{m} \frac{RB_0 b_{2m-1}}{(B\rho)} \left(\frac{\beta}{\beta_0}\right)^m \frac{\mathcal{A}^{2m}}{2^{2m} m} \binom{2m}{\ell} \frac{\cos 2(m-\ell)(Q+a)}{\delta_{\ell m}+1}$$

$$+ \sum_{m=1}^{\infty} \sum_{\ell=0}^{m} \frac{RB_0 b_{2m}}{(B\rho)} \left(\frac{\beta}{\beta_0}\right)^{m+\frac{1}{2}} \frac{\mathcal{A}^{2m+1}}{2^{2m}(2m+1)} \binom{2m+1}{\ell} \cos(2m-2\ell+1)(Q+a) \ . \tag{4.10}$$

The change of I with respect to θ is

$$\frac{dI}{d\theta} = -\frac{\partial \Delta H}{\partial a} = \sum_{m=1}^{\infty} \sum_{\ell=0}^{m-1} \frac{RB_0 b_{2m-1}}{(B\rho)} \left(\frac{\beta}{\beta_0}\right)^m \frac{\mathcal{A}^{2m}(m-\ell)}{2^{2m-1} m} \binom{2m}{\ell} \sin 2(m-\ell)(Q+a)$$

$$+ \sum_{m=1}^{\infty} \sum_{\ell=0}^{m} \frac{RB_0 b_{2m}}{(B\rho)} \left(\frac{\beta}{\beta_0}\right)^{m+\frac{1}{2}} \frac{\mathcal{A}^{2m+1}(2m-2\ell-1)}{2^{2m}(2m+1)} \binom{2m+1}{\ell} \sin(2m-2\ell+1)(Q+a) \ . \tag{4.11}$$

Because we are interested in the first-order deviation from the linear amplitude $\delta\mathcal{A}$, or the first-order deviation from the linear action δI, on the right-hand side of Eq. (4.10), I can be considered θ-independent and a can be obtained from the equation

$$\frac{da}{d\theta} = \frac{\partial H}{\partial I} \approx \nu \ , \tag{4.12}$$

or

$$a(\theta) = \nu\theta + \varphi \ , \tag{4.13}$$

where φ is an initial phase. Then, the integration of Eq. (4.10) can be performed easily. Noting that $Q(\theta)$ is periodic in θ, we obtain

$$\delta I = -\sum_{m=1}^{\infty} \sum_{\ell=0}^{m-1} \int_{\theta}^{\theta+2\pi} d\theta' \frac{RB_0 b_{2m-1}/(B\rho)}{\sin 2(m-\ell)\pi\nu} \left(\frac{\beta}{\beta_0}\right)^m \frac{\mathcal{A}^{2m}(m-\ell)}{2^{2m} m} \binom{2m}{\ell} \times$$

$$\times \cos 2(m-\ell)(Q+a-\pi\nu) - \sum_{m=1}^{\infty} \sum_{\ell=0}^{m} \int_{\theta}^{\theta+2\pi} d\theta' \frac{RB_0 b_{2m}/(B\rho)}{\sin(2m-2\ell+1)\pi\nu} \times$$

$$\times \left(\frac{\beta}{\beta_0}\right)^{m+\frac{1}{2}} \frac{\mathcal{A}^{2m+1}(2m-2\ell+1)}{2^{2m+1}(2m+1)} \binom{2m+1}{\ell} \cos(2m-2\ell+1)(Q+a-\pi\nu) \,. \tag{4.14}$$

Taking the thin lens approximation and defining the multipole of lengths $L \to 0$ and strengths as

$$\mathcal{S}_k^{(2m-1)} = \lim_{L\to 0} \left[\frac{B_0 b_{2m-1}}{(B\rho)} \left(\frac{\beta}{\beta_0}\right)^m \beta_0 L\right]_k \qquad \text{for the } 4m\text{-th multipole}$$

$$\mathcal{S}_k^{(2m)} = \lim_{L\to 0} \left[\frac{B_0 b_{2m}}{(B\rho)} \left(\frac{\beta}{\beta_0}\right)^{m+\frac{1}{2}} \beta_0 L\right]_k \qquad \text{for the } (4m+2)\text{-th multipole}\,, \tag{4.15}$$

Eq. (4.13) can be rewritten as

$$\frac{\delta\mathcal{A}}{\mathcal{A}} = -\sum_{m=1}^{\infty}\sum_{\ell=0}^{m-1}\sum_k \frac{\mathcal{A}^{2m-2}\mathcal{S}_k^{(2m-1)}}{\sin 2(m-\ell)\pi\nu} \frac{m-\ell}{2^{2m}m} \binom{2m}{\ell} \cos 2(m-\ell)(\psi_k'-\psi-\pi\nu+\varphi)$$

$$-\sum_{m=1}^{\infty}\sum_{\ell=0}^{m}\sum_k \frac{\mathcal{A}^{2m-1}\mathcal{S}_k^{(2m)}}{\sin(2m-2\ell+1)\pi\nu} \frac{2m-2\ell+1}{2^{2m+1}(2m+1)} \binom{2m+1}{\ell} \cos(2m-2\ell+1)(\psi_k'-\psi-\pi\nu+\varphi)\,, \tag{4.16}$$

where the summation over k is for all the multipoles around the ring and we have used ψ_k', the modified phase location of the multipole as defined in Eq. (2.6).

We notice from Eq. (4.16) that multipoles of different orders can have the same cosine dependence and they will be mixed. Thus, it will be better to sum up $p = m-\ell$ instead. Then, Eq. (4.15) becomes

$$\begin{aligned}\frac{\delta\mathcal{A}}{\mathcal{A}} \;=\; & -\sum_{p=1}^{\infty}\sum_k\sum_{m=p}^{\infty} \frac{\mathcal{A}^{2m-2}\mathcal{S}_k^{(2m-1)} f_p^{(2m-1)}}{\sin 2p\pi\nu} \cos 2p(\psi_k'-\psi-\pi\nu+\varphi) \\ & -\sum_{p=0}^{\infty}\sum_k{\sum_{m=p}^{\infty}}' \frac{\mathcal{A}^{2m-1}\mathcal{S}_k^{(2m)} f_p^{(2m)}}{\sin(2p+1)\pi\nu} \cos(2p+1)(\psi_k'-\psi-\pi\nu+\varphi)\,,\end{aligned} \tag{4.17}$$

with $f_p^{(2m-1)}$ and $f_p^{(2m)}$ defined as

$$f_p^{(2m-1)} = \frac{p}{2^{2m}m} \binom{2m}{m-p} \qquad \text{for the } 4m\text{-th multipoles}$$

$$f_p^{(2m)} = \frac{2p+1}{2^{2m+1}(2m+1)} \binom{2m+1}{m-p} \qquad \text{for the } (4m+2)\text{-th multipoles}\,. \tag{4.18}$$

The $'$ in the summation implies that $m = 0$ is excluded. Finally, an average over the initial phase φ is performed to obtain the smear S defined in Eq. (2.9). We get

$$S^2 = \frac{1}{2}\sum_{p=1}^{\infty}\left|\sum_k \sum_{m=p}^{\infty} \frac{\mathcal{A}^{2m-2}\mathcal{S}_k^{(2m-1)} f_p^{(2m-1)}}{\sin 2p\pi\nu} e^{i2p\psi_k'}\right|^2$$

$$+ \frac{1}{2}\sum_{p=0}^{\infty}\left|\sum_k \sum_{m=p}^{\infty}{}' \frac{\mathcal{A}^{2m-1}\mathcal{S}_k^{(2m)} f_p^{(2m)}}{\sin(2p+1)\pi\nu} e^{i(2p+1)\psi_k'}\right|^2 . \tag{4.19}$$

If the higher multipoles discussed here are all random in nature, the square of the smear can be simplified to

$$S^2 = \frac{1}{2}\sum_{p=1}^{\infty}\sum_{m=p}^{\infty}\left[\frac{\mathcal{A}^{2m-2} f_p^{(2m-1)}}{\sin 2p\pi\nu}\right]^2 \sum_k \langle \mathcal{S}_k^{(2m-1)\,2}\rangle$$

$$+ \frac{1}{2}\sum_{p=0}^{\infty}\sum_{m=p}^{\infty}{}'\left[\frac{\mathcal{A}^{2m-1} f_p^{(2m)}}{\sin(2p+1)\pi\nu}\right]^2 \sum_k \langle \mathcal{S}_k^{(2m)\,2}\rangle , \tag{4.20}$$

where $\langle \mathcal{S}_k^{(2m-1)\,2}\rangle$ and $\langle \mathcal{S}_k^{(2m)\,2}\rangle$ are the variances of the $4m$-th and $(4m+2)$-th random multipole errors respectively.

V. APPLICATIONS

1. Beam Dynamics Experiment E778

Experiment E778 performed in the Fermilab Tevatron, studied the nonlinear dynamics of transverse particle oscillations.[14] Nonlinearities were introduced in the Tevatron by sixteen special sextupoles. A low emittance proton beam was injected into the Tevatron. The sextupoles were then ramped up to their final setting. The injection kicker was used to induce a coherent betatron oscillation. The centroid beam position was recorded in each of two adjacent beam position monitors. From this information the phase space motion could be tracked[15] and the smear could be extracted.

The smear was one of the parameters used to characterize deviation from linear behavior. Measurements were repeated over a wide range of conditions. Specifically, measurements were made at various values of the sextupole excitation, the horizontal tune, the kicker strength and the beam emittance. The sextupole excitation varied from 0 to 50 amperes in steps of 5 amperes. The horizontal tune assumed 5 different values from 19.38 to 19.42 in steps of .01 while the vertical tune was set to 19.46. The kicker strength was 5, 8 and 10 kilovolts corresponding to 2.25, 3.8 and 4.5 mm in amplitude. Finally, measurements were taken at two different ranges of the horizontal emittance. At the low range the emittance varied from 1.5π to 3.7π mm-mrad while at the high range it was between 7.8π and 10.9π mm-mrad.

Single and multiparticle tracking calculations were done to simulate the above experimental conditions and the smear was extracted from these calculations. It was then compared with the smear extracted from the experimental data. In order to demonstrate the agreement between experimental and simulated data for the smear we display Fig. (2). Here the smear is plotted against the sextupole excitation, for five tune values (19.38 to 19.42). The three curves in each of the five plots correspond to the three different kicker strengths. The dashed lines represent prediction from tracking calculations while the solid lines correspond to the experimental data.

Next the smear due to the sixteen special sextupoles was calculated using Eq. (2.12) and compared with both single particle tracking calculations and experimental data. Fig. (3) displays the comparison between the prediction of perturbation theory (solid

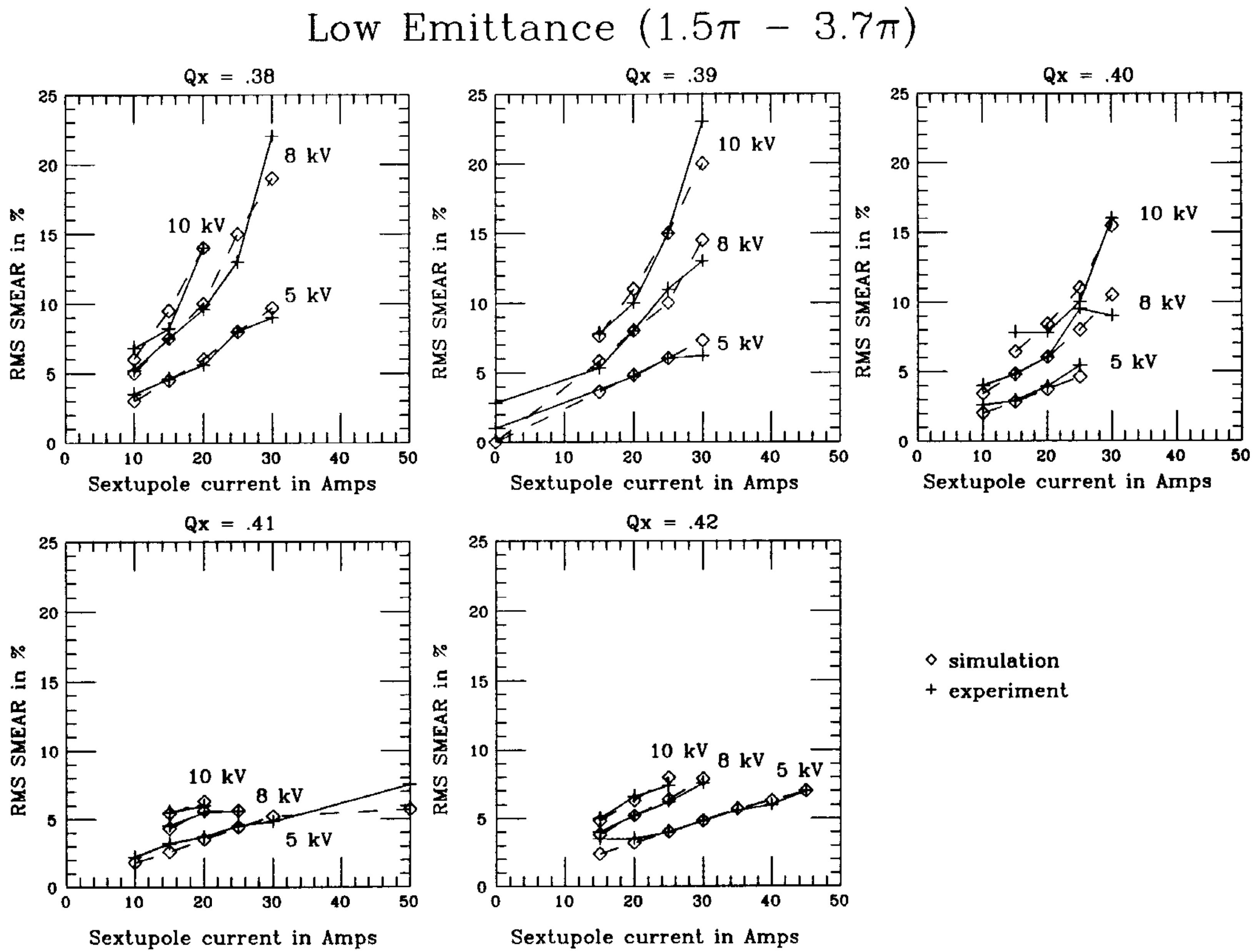

Figure 2: Smear versus sextupole excitation, for low emittance data. Comparison between experimental observations and tracking calculations.

line) and experimental data for a low emittance beam (crosses). Again the three different lines in each of the five plots correspond to 5, 8 and 10 kV of the kicker strength. The agreement between observation and prediction is very good especially in the low current – low kicker amplitude regime. The deterioration of the agreement observed at higher currents and kick amplitudes is due to the fact that nonlinearities are too strong to be handled perturbatively. Finally, Fig. (4) displays the comparison between perturbative calculations and single particle tracking predictions. The previous comments can presumably explain any disagreement between these two methods of predicting the smear.

2. Smear Due to Multipole Errors in the Tevatron Dipoles

2a. Random Errors

Each one of the 776 Tevatron dipoles is characterized by a set of harmonic coefficients (a_n, b_n): the normal and skew multipole coefficients of order $2(n+1)$. The mean value of each multipole component calculated over the number of dipoles constitutes the so-called systematic error while the rms value constitutes the random error of the particular multipole. Table I displays the values of both the systematic and random multipole errors for the Tevatron dipoles,[16] up to the 14-pole. These values correspond to a magnetic field of 4000 amperes. The units of b_n are 10^{-4} cm^{-n}, and the values are 10^{-4} of the integrated dipole field at 1 cm radius. In the following we shall use Eq. (4.20) to calculate the smear due to all higher multipoles of the Tevatron dipoles, assuming they are random in nature. Since the highest multipole with a non-zero coefficient is the 14-pole ($n = 6$), Eq. (4.20) becomes

$$S^2 = \frac{1}{2}\sum_{p=1}^{3}\sum_{m=p}^{3}{}' \left[\frac{\mathcal{A}^{2m-2} f_p^{(2m-1)}}{\sin 2p\pi\nu}\right]^2 \sum_{k=1}^{776}\langle \mathcal{S}_k^{(2m-1)\,2}\rangle$$

$$+ \frac{1}{2}\sum_{p=0}^{3}\sum_{m=p}^{3}{}' \left[\frac{\mathcal{A}^{2m-1} f_p^{(2m)}}{\sin(2p+1)\pi\nu}\right]^2 \sum_{k=1}^{776}\langle \mathcal{S}_k^{(2m)\,2}\rangle \,, \tag{5.1}$$

where the prime in the first sum implies that the $m = 1$ term is excluded (since b_1 is the normal quadrupole field) while in the second sum the prime implies that the $m = 0$

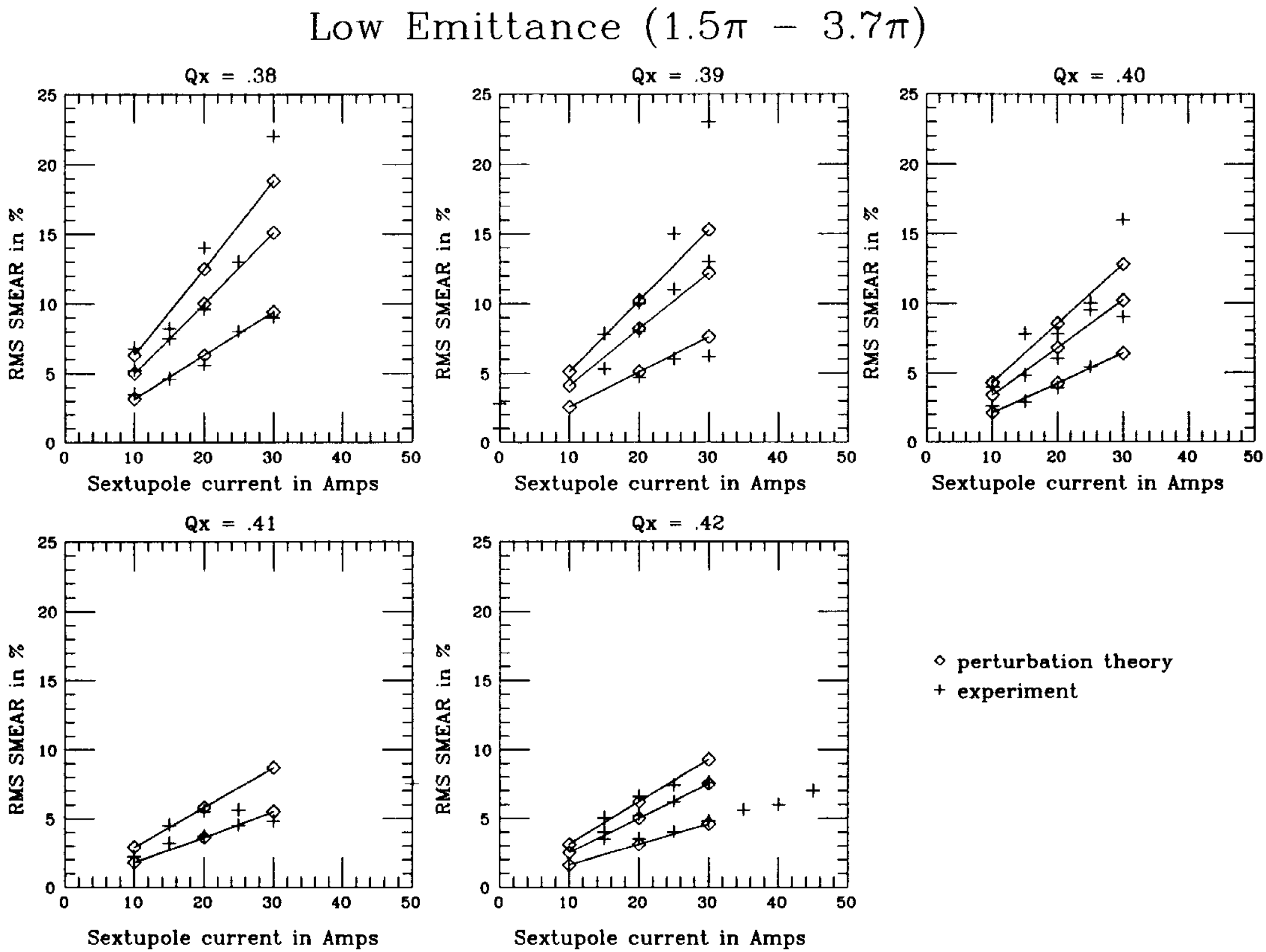

Figure 3: Smear versus sextupole excitation. Comparison between experimental observations and predictions from perturbation theory.

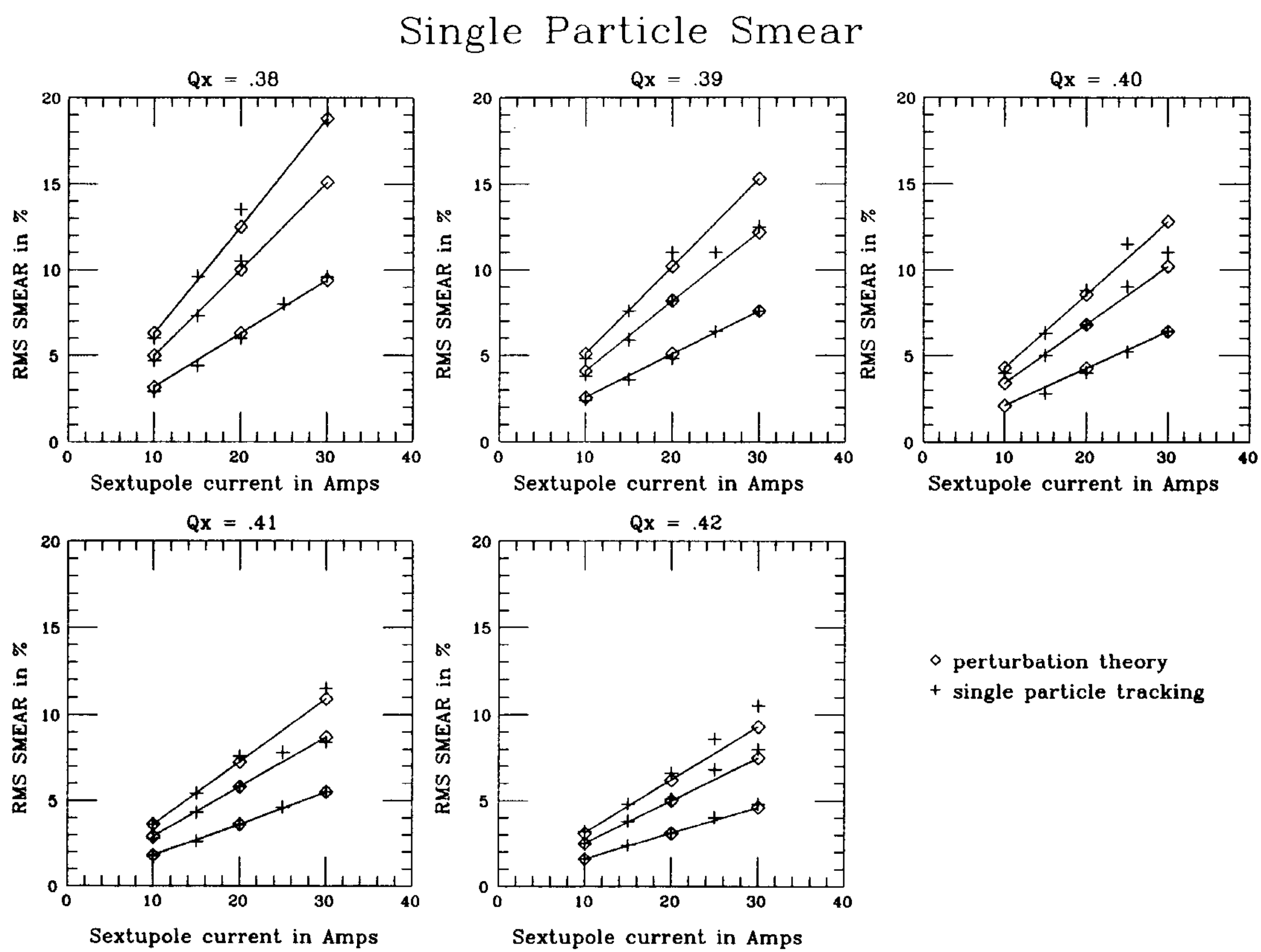

Figure 4: Smear versus sextupole excitation. Comparison between perturbative calculations and single particle tracking predictions.

mult. coeff.	multipole	System. errors	Random errors
b_1	normal quadrupole	0.035	0.189
b_2	normal sextupole	0.1	0.484
b_3	normal octupole	-0.014	0.047
b_4	normal decapole	-0.014	0.032
b_5	normal dodecapole	–	0.003
b_6	normal fourteen-pole	0.020	0.002

Table I: The constants b_n are the systematic and random multipole errors in the Tevatron dipoles, expressed in units of 10^{-4} cm^{-n}.

term is excluded (normal dipole field). The variance of the random multipole errors is given by

$$\langle S_k^{(l)^2} \rangle = \left\{ \left(\frac{B_0 \beta_0}{(B\rho)} L \right)_k b_l \right\}^2 \tag{5.2}$$

since we chose $\beta_0 = \beta$.

For the Tevatron B_0=4.4 Tesla, β_0=100 m, the dipole length is $L = 6.12$ m and the magnetic rigidity $B\rho$=10/3×900 Tesla-meters. The b_l's are given in Table I. This calculation is done for a betatron amplitude of $\mathcal{A}$=5 mm and for a tune of 19.23. Using these parameters, the result for the smear in the Tevatron due to random errors in the dipole magnets is

$$S_{\text{Tev/ran}} = 1.04\ \%. \tag{5.3}$$

This result is in agreement with observations done as part of the E778 experiment in order to demonstrate the linearity of the Tevatron.[17]

2b. Systematic Errors

In order to calculate the smear due to the systematic errors in the Tevatron dipoles we use Eq. (4.19). In the presence of the errors shown in Table I, Eq. (4.19) becomes

$$S^2 = \frac{1}{2}\sum_{p=1}^{5}\left|\sum_{k=1}^{776}\sum_{m=p}^{5}{}' \frac{\mathcal{A}^{2m-2}\mathcal{S}_k^{(2m-1)} f_p^{(2m-1)}}{\sin 2p\pi\nu} e^{i2p\psi_k'}\right|^2 + \frac{1}{2}\sum_{p=0}^{4}\left|\sum_{k=1}^{776}\sum_{m=p}^{4}{}' \frac{\mathcal{A}^{2m-1}\mathcal{S}_k^{(2m)} f_p^{(2m)}}{\sin(2p+1)\pi\nu} e^{i(2p+1)\psi_k'}\right|^2 , \tag{5.4}$$

or

$$S^2 = \frac{1}{2}\sum_{p=1}^{5}\frac{1}{\sin^2 2p\pi\nu}\left|\sum_{k=1}^{776} e^{i2p\psi_k'}\sum_{m=p}^{5}{}' \left[\mathcal{A}^{2m-2}\mathcal{S}_k^{(2m-1)} f_p^{(2m-1)}\right]\right|^2 + \frac{1}{2}\sum_{p=0}^{4}\frac{1}{\sin^2(2p+1)\pi\nu}\left|\sum_{k=1}^{776} e^{i(2p+1)\psi_k'}\sum_{m=p}^{4}{}' \left[\mathcal{A}^{2m-1}\mathcal{S}_k^{(2m)} f_p^{(2m)}\right]\right|^2 , \tag{5.5}$$

where again the prime in the first and second sum means that $m = 1$ and $m = 0$ respectively are excluded.

If we consider a simplified Tevatron lattice which basically consists of a series of FODO cells with the 776 dipoles equally spaced from each other around the machine then the k-th dipole is at a location with phase advance

$$\psi_k = 2\pi \times 19.23 \times \frac{k}{776} \tag{5.6}$$

for a tune of ν=19.23. In this case one can easily perform the summations

$$\sum_{k=1}^{776} e^{i2p\psi_k'} \tag{5.7}$$

and

$$\sum_{k=1}^{776} e^{i(2p+1)\psi_k'} \tag{5.8}$$

as finite geometric progressions. Indeed

$$\sum_{k=1}^{776} e^{i2p\psi_k'} = \sum_{k=1}^{776} e^{2ip\tau k} = \frac{e^{i4p\pi\nu}-1}{1-e^{-2ip\tau}} , \tag{5.9}$$

where

$$r = \frac{2\pi\nu}{776} \,. \tag{5.10}$$

Also notice that the multipole strengths $\mathcal{S}_k$ are the same for all k and hence Eq. (5.5) is simplified to

$$\begin{aligned} S^2 = \frac{1}{2}\sum_{p=1}^{5} \frac{1}{\sin^2 2p\pi\nu} \left|\frac{e^{i4p\pi\nu}-1}{1-e^{-2ipr}}\right| \left|\sum_{m=p}^{5}{}' \mathcal{A}^{2m-2}\mathcal{S}^{(2m-1)} f_p^{(2m-1)}\right|^2 \\ + \frac{1}{2}\sum_{p=0}^{4} \frac{1}{\sin^2 (2p+1)\pi\nu} \left|\frac{e^{i2(2p+1)\pi\nu}-1}{1-e^{-i(2p+1)r}}\right| \left|\sum_{m=p}^{4}{}' \mathcal{A}^{2m-1}\mathcal{S}^{(2m)} f_p^{(2m)}\right|^2 . \end{aligned} \tag{5.11}$$

Using the values specified above, the result for the smear in the Tevatron due to systematic errors in the dipole magnets, is

$$S_{\mathrm{Tev/syst}} \;=\; 0.80\%. \tag{5.12}$$

3. Smear Due to Multipole Errors in the SSC Dipoles Before and After the Insertion of Correction Elements

The SSC dipoles are expected to have higher order multipole components which are given in Table II below. The units of the harmonic coefficients b_n are again 10^{-4} cm^{-n}. Though both random and systematic errors are present, the random errors are the principal source of the smear. Hence the linear aperture of the SSC is dominated by the random multipole components of the dipole magnets. Various correction schemes have been devised to compensate for the random errors anticipated in the SSC. It appears[9] that the so-called three lumped correction scheme due to Neuffer[8] is the most effective one.

In the following we shall calculate the smear in the SSC due to random and/or systematic errors in the dipoles using the formalism developed above. Then, we shall introduce correction elements according to the three lumped correction scheme and recalculate the smear for various situations. The purpose of this exercise is to demonstrate that one can predict the smear of a fairly complicated lattice from analytical methods without resorting to extensive tracking.

mult. coeff.	multipole	System. errors	Random errors
b_1	normal quadrupole	0.2	0.7
b_2	normal sextupole	1.0	2.0
b_3	normal octupole	0.1	0.3
b_4	normal decapole	0.2	0.7
b_5	normal dodecapole	0.02	0.1
b_6	normal fourteen-pole	0.04	0.2
b_7	normal sixteen-pole	0.06	0.2
b_8	normal eighteen-pole	0.1	0.1

Table II: The constants b_n are the systematic and random multipole errors in the SSC dipoles, expressed in units of 10^{-4} cm^{-n}.

For this calculation we consider the 'arcs only' SSC lattice which consists solely of 320 identical cells, which in turn contain only the dipoles and the lumped correction elements. There are 12 dipoles per cell. The horizontal tune of the lattice was chosen to be $\nu_x = 81.285$. The dipole errors included in this study are the random and/or systematic sextupole, octupole and decapole errors. Their skew components are not included.

All the calculations in this section will be done with the use of Eq. (4.19), which, in the presence of b_2, b_3 and b_4 only, for the SSC lattice becomes

$$S^2 = \frac{1}{2}\sum_{p=1}^{2}\left|\sum_k \sum_{m=p}^{2}{}' \frac{A^{2m-2} S_k^{(2m-1)} f_p^{(2m-1)}}{\sin 2p\pi\nu} e^{i2p\psi_k'}\right|^2$$
$$+\frac{1}{2}\sum_{p=0}^{2}\left|\sum_k \sum_{m=p}^{2}{}' \frac{A^{2m-1} S_k^{(2m)} f_p^{(2m)}}{\sin(2p+1)\pi\nu} e^{i(2p+1)\psi_k'}\right|^2 . \qquad (5.13)$$

The basic idea of Neuffer's scheme is the use of three lumped correctors per half

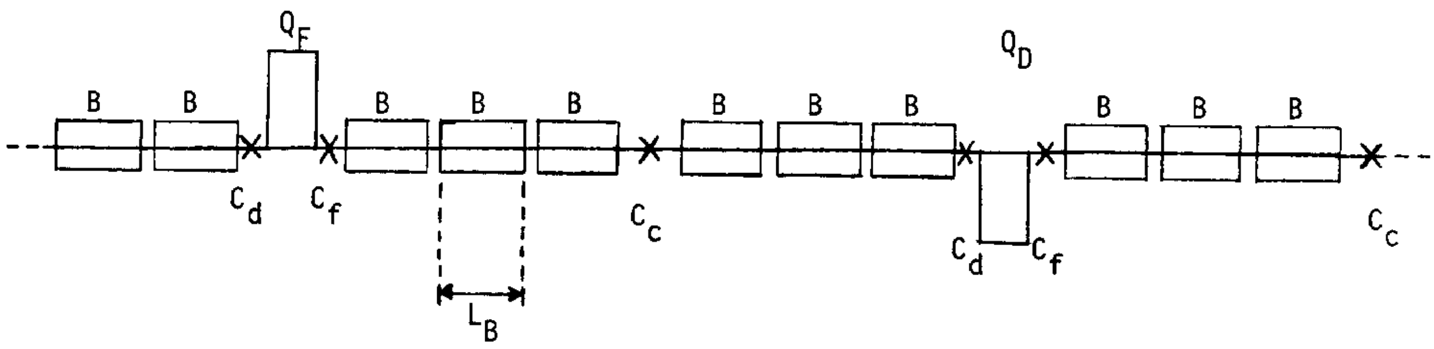

Figure 5: Neuffer's or 'three-lumped-correctors' scheme. B denotes the dipole magnets, Q denotes the quad, C_c the middle corrector and $C_{f,d}$ the correctors before and after each quad.

cell, one on each side of the quadrupole, $C_{f,d}$, and one in the middle of the half cell, C_c, as Fig. (5) demonstrates. The integral strengths of the three correctors are given by[8,10]

$$C_fL_f = \frac{L_B}{108}[-83\alpha_1 - 41\alpha_2 - 11\alpha_3 + 7\alpha_4 + 13\alpha_5 + 7\alpha_6] , \tag{5.14}$$

$$C_cL_c = -\frac{L_B}{27}[8\alpha_1 + 20\alpha_2 + 26\alpha_3 + 26\alpha_4 + 20\alpha_5 + 8\alpha_6] , \tag{5.15}$$

$$C_dL_d = \frac{L_B}{108}[7\alpha_1 + 13\alpha_2 + 7\alpha_3 - 11\alpha_4 - 41\alpha_5 - 83\alpha_6] . \tag{5.16}$$

Here α_i is the multipole error of the i-th dipole magnet.

Next, we are going to describe the calculation we performed in detail so one can easily reproduce it. Let us suppose we want to compensate for b_2, b_3 and b_4. First, we randomly generate three gaussian distributions of errors for each of the three multipoles. Each distribution constitutes 3840 errors (one for each SSC dipole magnet). The mean and the standard deviation of these distributions are given in Table II. Then we consider the earlier described simplified SSC lattice whose basic cell consists of

$$C_fBBBC_cBBBC_dC_fBBBC_cBBBC_d. \tag{5.17}$$

The virtue of Neuffer's correction scheme lies in the fact that one may ignore the quadrupoles and combine the two correctors C_f and C_d together into one with integral strength equal to the sum of the two. Notice that the strengths of the first three correctors on the left of (5.17) are evaluated via the errors of the first six dipoles on the left, while the other three correctors are evaluated via the errors of the six dipoles on the right.

All these elements are considered as thin lenses. The correctors are located in the middle of the interval between two successive dipoles. Thus there are four contributions to the summation over k in Eq. (4.19). Specifically Eq. (5.13) becomes

$$\begin{aligned} S^2 = \frac{1}{2}\sum_{p=1}^{2}\Bigg| \sum_{m=p}^{2} \frac{\mathcal{A}^{2m-2} f_p^{(2m-1)}}{\sin 2p\pi\nu} \times \\ \left[\sum_{k=1}^{3840} \mathcal{S}_{Bk}^{(2m-1)} e^{i2p\psi'_{Bk}} + \sum_{k=1}^{640} \mathcal{S}_{ck}^{(2m-1)} e^{i2p\psi'_{ck}} + \sum_{k=1}^{640} \mathcal{S}_{dk}^{(2m-1)} e^{i2p\psi'_{dk}} + \sum_{k=1}^{640} \mathcal{S}_{fk}^{(2m-1)} e^{i2p\psi'_{fk}}\right]\Bigg|^2 \\ + \frac{1}{2}\sum_{p=0}^{2}\Bigg| \sum_{m=p}^{2} \frac{\mathcal{A}^{2m-1} f_p^{(2m)}}{\sin(2p+1)\pi\nu} \times \left[\sum_{k=1}^{3840} \mathcal{S}_{Bk}^{(2m)} e^{i(2p+1)\psi'_{Bk}} \right. \\ \left. + \sum_{k=1}^{640} \mathcal{S}_{ck}^{(2m)} e^{i(2p+1)\psi'_{ck}} + \sum_{k=1}^{640} \mathcal{S}_{dk}^{(2m)} e^{i(2p+1)\psi'_{dk}} + \sum_{k=1}^{640} \mathcal{S}_{fk}^{(2m)} e^{i(2p+1)\psi'_{fk}}\right]\Bigg|^2 . \end{aligned} \tag{5.18}$$

The strength of the l-th multipole of the k-th dipole B, $\mathcal{S}_{Bk}^{(l)}$, is given by

$$\mathcal{S}_{Bk}^{(l)} = \frac{B_0 \alpha_k^{(l)}}{(B\rho)} \left(\frac{\beta}{\beta_0}\right)^{(l+1)/2} \beta_0 L \ , \tag{5.19}$$

where $\alpha_k^{(l)}$ is the error of the k-th dipole magnet. Similarly, the strength of the l-th multipole of the k-th corrector C_j, is

$$\mathcal{S}_{jk}^{(l)} = \frac{B_0\beta_0}{(B\rho)}(C_j L_j) \ , \tag{5.20}$$

where the subscript j stands for c, f and d. Also $C_c L_c$ is given by Eq. (5.15), $C_f L_f$ is given by Eq. (5.14) and $C_d L_d$ is given by Eq. (5.16).

For the SSC lattice, the numerical values of the quantities defined above are: at the energy of 20 TeV, B_0 is 6.6 Tesla. The dipole length L_B is 16.54 meters and the maximum beta β_0 is 332.0 meters. In our calculation we assumed $\beta = \beta_0$. The amplitude was 5 mm.

	Smear (%)
No correction. b_2, b_3, b_4 random and systematic present.	7.09 ± 2.53
No correction. Random b_2, b_3, b_4 present.	7.07 ± 2.50
No correction. Systematic b_2, b_3, b_4 present.	0.39 ± 0.00
Correct random and systematic b_2, b_3, b_4. Random and systematic b_2, b_3, b_4 present.	0.43 ± 0.22
Correct random b_2, b_3, b_4. Random b_2, b_3, b_4 present.	0.43 ± 0.22
Correct random b_2. Random b_2, b_3, b_4 present.	0.71 ± 0.25
Correct systematic b_2, b_3, b_4. Systematic b_2, b_3, b_4 present.	0.0009 ± 0.00

Table III: Summary of the results of the analytic computation of the smear in the SSC, with and without correction elements.

The results of our computation are summarized in Table III. The value of the smear quoted is the average over 100 seeds and the standard deviation of the mean is quoted as the uncertainty in the smear. It is worth pointing out that the value of the smear fluctuates by a large amount depending on the seed one uses. It is important that this fact is taken into account in the design of a real machine. If only the mean value of the smear is used as a criterion for the determination of the linear aperture (apart from the tuneshift), the good field region may turn out not to be sufficient for safe operation. One should allow the smear to vary as much as say two standard deviations away from the mean value, within the good field region.

The general conclusion is that the three lumped correction scheme is very effective in compensating for the random errors. Moreover, this particular scheme is also very effective in correcting for the systematic errors. This is expected, since it was initially developed to correct for the systematic components, and only later it was shown that it can be used to correct the random errors as well.

Another observation is that the horizontal smear (in the absence of skew multipoles) is dominated by the sextupole component b_2. The effects of b_3 and b_4 are rather weak.

APPENDIX

The horizontal motion of a beam particle is described by the Hamiltonian[12,13]

$$H_1 = \frac{1}{2}\left[P_x^2 + K_x(s)X^2\right] + \sum_{n=1}^{\infty} \frac{B_0 b_n}{(B\rho)} \frac{X^{n+1}}{n+1} , \tag{A.1}$$

where X is the horizontal displacement, P_x is the canonical momentum, and K_x represents the field gradient of the normal quads. The independent variable s is the distance measured along some designed closed orbit. In below, the subscript x will be suppressed. We perform a canonical transformation into the Floquet space using the generation function

$$G_1(x, P; s) = -\left(\frac{\beta}{\beta_0}\right)^{1/2} xP + \frac{\beta' x^2}{4\beta_0} , \tag{A.2}$$

where the differentiation in β' is with respect to s. The Hamiltonian becomes

$$H_2 = \frac{R}{2\beta}\left[\beta_0 p^2 + \frac{x^2}{\beta_0}\right] + \sum_{n=1}^{\infty} \frac{RB_0 b_n}{(B\rho)} \left(\frac{\beta}{\beta_0}\right)^{\frac{n+1}{2}} \frac{x^{n+1}}{n+1} . \tag{A.3}$$

Here, the independent variable has been changed to $\theta = s/R$, where R is the average radius of the designed closed orbit. Use has been made of the relation

$$2\beta\beta'' - \beta'^2 + 4K\beta^2 = 4 , \tag{A.4}$$

which defines the beta-function. The horizontal displacement X is related to the transformed displacement x by

$$x = \left(\frac{\beta_0}{\beta}\right)^{1/2} X . \tag{A.5}$$

The first term of the Hamiltonian is now solved exactly by canonical transformation to the action-angle variables I and a, while the second term is treated as a perturbation. The generating function is

$$G_2(a, p; \theta) = \frac{1}{2}\beta_0 p^2 \cot[Q(\theta) + a] . \tag{A.6}$$

The canonical variables are transformed to

$$\begin{aligned} x &= (2I\beta_0)^{1/2} \cos[Q(\theta) + a] , \\ \beta_0 p &= -(2I\beta_0)^{1/2} \sin[Q(\theta) + a] , \end{aligned} \tag{A.7}$$

where $Q(\theta)$ is given by Eq. (4.8). The transformed Hamiltonian is given by Eq. (4.5).

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